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Stability of matrices with negative diagonal submatrices

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Abstract

This paper discusses stability conditions for matrices that determine the homogeneous dynamics of systems of linear second-order differential equations. In particular, we focus on situations in which these matrices have a negative diagonal submatrix. We present a number of theorems that provide conditions which are sufficient for either stability or instability of such matrices. In order to discuss the instability theorems and unify them with earlier results, we introduce the concept of the dominant diagonal number of a matrix.

Keywords: Stability; Dominant diagonal

1. Introduction

Consider the system of second-order differential equations $\ddot{y} = A\dot{y} + By$, where y represents an $n \times 1$ vector of variables, $\ddot{y}$ and $\dot{y}$ are respectively its second-order and first-order time derivative, and $A \equiv (a_{ij})$ and $B \equiv (b_{ij})$ are $n \times n$ real matrices. As is well known, the dynamics of this system are determined by the eigenvalues of the following $2n \times 2n$ matrix

$$C = \begin{bmatrix} A & B \\ I & O \end{bmatrix}, \tag{1}$$

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which is obtained by rewriting the second-order system in the obvious way as a system of first-order equations. Here I is an $n \times n$ identity matrix, and O is an $n \times n$ matrix of zeros. The system is asymptotically stable if and only if all eigenvalues of the matrix C have a negative real part. In that case we call C a stable matrix. Asymptotic stability means that any solution of the second-order system converges towards the equilibrium vector with all elements equal to zero, See e.g. [2].

In [3] we have investigated the stability properties of the matrix C in terms of conditions on the matrices A and B. In particular, we focused on the situation in which the diagonal elements of A and B are negative. Generalizing the concept of a matrix with a (negative) dominant diagonal, we introduced the definition of a matrix that has a (negative) dominant diagonal with a given strength factor $0 < \tau < 1$. We showed that if the matrices A and B have a negative dominant diagonal with certain deliberately chosen strength factors τ which are 'small enough' (i.e., which are sufficiently smaller than unity), then C is a stable matrix. We also mentioned a number of applications in which stability analysis of the matrix C is relevant. See [3] for details.

This paper extends the analysis of [3] and presents new stability results for the matrix C. In particular, we will analyse cases here in which again the diagonal elements of A and B are negative, and moreover either A or B is a diagonal matrix. In Section 2 we introduce and elaborate on a new concept, i.e., the so-called *dominant diagonal number* of a matrix. In Section 3 we give two lemmas which are useful later on. In Section 4 we present a number of stability theorems for the matrix C. We show that these results are stronger than and/or extend related stability results given in [3]. In Section 5 we give two instability results for the matrix C for the cases in which one of the matrices A and B is Metzlerian while the other is a negative diagonal matrix. In order to prove the instability we demonstrate in each case that C has a real nonnegative eigenvalue. Incidentally, we also show that these real nonnegative eigenvalues are accompanied by a semipositive eigenvector. Finally, we point out that we can unify the instability theorems and relate them to the results of [3] by using the concept of the dominant diagonal number of a matrix as introduced in Section 2. We briefly conclude in Section 6.

2. The dominant diagonal number of a matrix

To begin with, we introduce some notation and recall some basic definitions. Let x be an arbitrary real vector. Then $x > 0$ (x positive) means that all elements of x are positive; $x \geqslant 0$ (x semipositive) means that all elements are nonnegative while at least one element is positive; $x \geqq 0$ (x nonnegative) means that all elements are nonnegative. Similar notation is used for matrices.

Next, let $H \equiv (h_{ij})$ be an arbitrary real $n \times n$ matrix. The notation $H = \mathrm{diag}(h_{11}, \ldots, h_{nn})$ means that H is an $n \times n$ diagonal matrix with diagonal entries as listed between the brackets. If, in addition, all diagonal entries are negative (positive), then

H is called a negative (positive) diagonal matrix. The matrix H is called Metzlerian if $h_{ii} < 0$ for all i and $h_{ij} \geqq 0$ for all $i \neq j$.

An arbitrary (possibly complex) $n \times n$ matrix $H \equiv (h_{ij})$ has a dominant diagonal, which will be abbreviated as dd, if there exist positive scalars (weights) m_i, $i = 1, \ldots, n$, such that $m_i|h_{ii}| > \sum_{j\neq i} m_j|h_{ij}|$ for all i. If, in addition, $h_{ii} < 0$ for all i, then H has a negative dominant diagonal, which will be abbreviated as ndd. We recall that H is stable in case it has an ndd. Furthermore, if H is a Metzlerian matrix, then H is stable if and only if H has an ndd. See e.g. [2].

Generalizing the concept of a dd, [3] defined that the matrix H has a dd with a *given strength factor* τ, where $0 < \tau < 1$, if there exist positive scalars m_i, $i = 1, \ldots, n$, such that $\tau m_i|h_{ii}| \geqq \sum_{j\neq i} m_j|h_{ij}|$ for all i. If, in addition, $h_{ii} < 0$ for all i, then H has an ndd with strength factor τ. We remark that if H has a dd with a given strength factor τ with $0 < \tau < 1$, then H also has a dd with any strength factor $\tau < \sigma < 1$, and moreover H has an (ordinary) dd. It is possible that H has a dd with some smaller strength factor $0 < \kappa < \tau$ as well, but that is not necessarily the case.

We now introduce the following definition which will be used in Section 5.

Definition 1. Consider a (possible complex) $n \times n$ matrix $H \equiv (h_{ij})$. The dominant diagonal number of H is the (unique) scalar $\sigma_H \geqq 0$ such that
(i) for all $0 \leqq \sigma \leqq \sigma_H$ there do *not* exist positive scalars m_i, $i = 1, \ldots, n$, such that $\sigma m_i|h_{ii}| > \sum_{j\neq i} m_j|h_{ij}|$ for all i;
(ii) for all $\sigma > \sigma_H$ there exist positive scalars m_i, $i = 1, \ldots, n$, such that $\sigma m_i|h_{ii}| > \sum_{j\neq i} m_j|h_{ij}|$ for all i.

Examining this definition, consider first the situation in which at least one of the diagonal entries of the matrix H is equal to zero. In that case, we have that $\sigma_H = \infty$. Second, take the situation in which all diagonal entries of H are nonzero. Now there is a unique finite σ_H. In order to illustrate this, we can define associated with such a matrix H the sets

$$S_H^1 = \Bigg\{\sigma > 0 \mid \nexists\, m_i > 0,\ i = 1, \ldots, n,\ \text{such that} \\ \sigma m_i|h_{ii}| > \sum_{j\neq i} m_j|h_{ij}| \text{ for all } i\Bigg\} \tag{2}$$

and

$$S_H^2 = \Bigg\{\sigma > 0 \mid \exists\, m_i > 0,\ i = 1, \ldots, n,\ \text{such that} \\ \sigma m_i|h_{ii}| > \sum_{j\neq i} m_j|h_{ij}| \text{ for all } i\Bigg\}. \tag{3}$$

Take an arbitrary scalar $\sigma^* > 0$. Then we must have either $\sigma^* \in S_H^1$ or $\sigma^* \in S_H^2$. Further, if $\sigma^* \in S_H^1$, then $\sigma \in S_H^1$ for all σ with $0 < \sigma < \sigma^*$. If $\sigma^* \in S_H^2$, then $\sigma \in S_H^2$ for all $\sigma > \sigma^*$. Now, we can distinguish between two possible cases, i.e.:

(1) $\sigma_H = 0$, which means that $S_H^1 = \emptyset$ and $S_H^2 = (0, \infty)$;

(2) $0 < \sigma_H < \infty$ and $S_H^1 = (0, \sigma_H]$ and $S_H^2 = (\sigma_H, \infty)$.

The case $0 < \sigma_H < \infty$, $S_H^1 = (0, \sigma_H)$ and $S_H^2 = [\sigma_H, \infty)$ cannot occur, because $\sigma_H \in S_H^2$ contradicts with (i) of Definition 1.

We remark that $\sigma_H = 0$ in case H is an upper (or lower) triangular matrix (or more particularly, a diagonal matrix) with nonzero diagonal entries. Conversely, if $\sigma_H = 0$ for a matrix H with nonzero diagonal entries, then there must exist an $n \times n$ permutation matrix G such that GHG' is an upper (or lower) triangular matrix. In order to understand this, define associated with the matrix H the scalar $h = \max_i\{|h_{ii}|\} > 0$ and the nonnegative matrix $\hat{H} \equiv (\hat{h}_{ij})$, with $\hat{h}_{ii} = 0$ for all i and $\hat{h}_{ij} = |h_{ij}|$ for all $i \neq j$. If $\sigma_H = 0$, then $S_H^2 = (0, \infty)$, which in turn means that for all $\sigma > 0$ there exist $m_i > 0$, $i = 1, \ldots, n$, such that $\sigma h m_i > \sum_{j \neq i} m_j |h_{ij}|$ for all i. Using [5, p. 393] we conclude from this that the Perron root of the matrix $\hat{H}$ is equal to zero. It then easily follows from [1, pp. 75, 76, 85] that there must exist a permutation matrix G such that $G\hat{H}G'$ is an upper (or lower) triangular matrix with all diagonal entries equal to zero. This implies that GHG' is an upper (or lower) triangular matrix with nonzero diagonal entries.

Further, we observe that $0 \leqq \sigma_H < 1$ if and only if H has an (ordinary) dd. In order to see this, suppose first that H has a dd. Then $1 \in S_H^2$, which implies that $0 \leqq \sigma_H < 1$. On the other hand, if $0 \leqq \sigma_H < 1$, then it follows from (ii) of Definition 1 that H must have a dd.

In order to relate Definition 1 to the analysis of [3], suppose that a matrix H with nonzero diagonal entries has a dd with a given strength factor $0 < \tau < 1$, as is defined in [3]. Recall that in that case there exist positive scalars m_i, $i = 1, \ldots, n$, such that $\tau m_i |h_{ii}| \geqq \sum_{j \neq i} m_j |h_{ij}|$ for all i. Using Definition 1, we obtain that the dominant diagonal number σ_H of H must satisfy $\sigma_H \leqq \tau$, i.e., the given strength factor τ constitutes an upper bound for the value of the dominant diagonal number of H.

Next, we present the following lemma which will be useful in Section 5.

Lemma 1. *Let $H \equiv (h_{ij})$ be a (possibly complex) $n \times n$ matrix with $h_{ii} < 0$ for all i. Let σ_H be the dominant diagonal number of H. We then have*:

(a) $0 \leqq \sigma_H < 1$ *if and only if there exists a positive diagonal matrix D such that $H + D$ has an ndd*;

(b) $\sigma_H > 1$ *if and only if there exists a negative diagonal matrix D such that $H + D$ has no ndd.*

Proof. (a) Suppose first that $0 \leqq \sigma_H < 1$. It is easy to see that then there must exist a scalar $\epsilon > 0$ and weights $m_i > 0$, $i = 1, \ldots, n$, such that $h_{ii} + \epsilon < 0$ and $m_i |h_{ii} + \epsilon| > \sum_{j \neq i} m_j |h_{ij}|$ for all i. The 'only if' part is established if we take $D = \mathrm{diag}(\epsilon, \ldots, \epsilon)$.

Now, suppose that for a given matrix $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$ with $d_{ii} > 0$ for all i, the matrix $H + D$ has an ndd with weights $m_i > 0$, $i = 1, \ldots, n$, i.e., $h_{ii} + d_{ii} < 0$ and $m_i|h_{ii} + d_{ii}| > \sum_{j \neq i} m_j|h_{ij}|$ for all i. Let us now define $\sigma_i = 1 + (d_{ii}/h_{ii}) > 0$ for all i, and $\bar{\sigma} = \max_i \sigma_i$. It then follows that $0 < \bar{\sigma} < 1$ and $\bar{\sigma} m_i |h_{ii}| \geqq m_i|\sigma_i h_{ii}| = m_i|h_{ii} + d_{ii}| > \sum_{j \neq i} m_j|h_{ij}|$ for all i. In other words, $0 \leqq \sigma_H < \bar{\sigma} < 1$, which completes the 'if' part.

(b) Suppose first that $\sigma_H > 1$. Define now $\hat{\sigma} = 1 + (\sigma_H - 1)/2 > 1$ and notice that $\hat{\sigma} \in S_H^1$. Next, define $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$ with $d_{ii} = (\hat{\sigma} - 1)/h_{ii} < 0$ for all i. We observe that $\hat{\sigma} \in S_H^1$ in fact means that the matrix $H + D$ has no ndd, which completes the 'only if' part.

Now, suppose that the matrix $H + D$ has no ndd, where D is a negative diagonal matrix. As a result, there do not exist weights $m_i > 0$, $i = 1, \ldots, n$, such that $m_i|h_{ii}(1 + (d_{ii}/h_{ii})| > \sum_{j \neq i} m_j|h_{ij}|$ for all i. Define next $\tilde{\sigma} = \min_i\{1 + (d_{ii}/h_{ii})\} > 1$ and observe that there do not exist weights $m_i > 0$, $i = 1, \ldots, n$, such that $\tilde{\sigma} m_i|h_{ii}| > \sum_{j \neq i} m_j|h_{ij}|$ for all i. In other words, $1 < \tilde{\sigma} \in S_H^1$, which in turn means that $\sigma_H \geqq \tilde{\sigma} > 1$, and we have established the 'if' part. □

3. Two useful lemmas

In this section we present two additional lemmas that will be useful in the sequel. We remark that Lemma 2 closely corresponds to [3, Lemma 1].

Lemma 2. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Then we have*:

(a) *μ is an eigenvalue of C if and only if it is a root of* $\det(G(\mu)) = 0$, *where $G(\mu) \equiv (g_{ij}^{\mu})$ is defined as the matrix $G(\mu) = \mu^2 I - \mu A - B$*;

(b) *If A and B have an ndd with the same weights $m_1, \ldots, m_n$, then all real eigenvalues of C are negative.*

Proof. (a) Let μ denote an eigenvalue of C, i.e., μ is a root of the characteristic equation $\det(\mu I_{2n\times 2n} - C) = 0$, where $I_{2n\times 2n}$ is a $2n \times 2n$ identity matrix. First, if $\mu \neq 0$, then the lower-right $n \times n$ submatrix of $(\mu I_{2n\times 2n} - C)$ is nonsingular. As a result, we can use a formula for the determinant of a partitioned matrix (see [2, p. 9]) and rewrite $\det(\mu I_{2n\times 2n} - C) = 0$ as $\det(G(\mu)) = 0$. Second, it is easy to see that $\mu = 0$ is an eigenvalue of C if and only if B is a singular matrix. This completes the proof of this part.

(b) See [3, Lemma 1, part (b)]. □

In order to present the next lemma, we need the following definition.

Definition 2. Let $Q \equiv (q_{ij})$ be a real $n \times n$ matrix with $q_{ii} > 0$ for all i and $q_{ij} \leqq 0$ for all $i \neq j$. Let $p_i(\mu) : M \to \mathbb{R}$ be a differentiable function for all $i = 1, \ldots, n$, where $M = (\mu_{\min}, \infty)$ and $\mu_{\min} \in \mathbb{R}$. Let there be a scalar $\hat{\mu} > \mu_{\min}$ such that

$p_i(\mu) \geqq 0$ as well as $p_i'(\mu) > 0$ for all $\mu \geqq \hat{\mu}$ and all i. We then define for $\mu \in M$ the matrices $P(\mu) \equiv \operatorname{diag}(p_1(\mu), \dots, p_n(\mu))$ and $Q(\mu) \equiv Q + P(\mu)$, and the set $H \equiv \{\mu \in M \mid Q^{-1}(\mu) \geqslant 0\}$.

We make the following three observations with respect to Definition 2. First, the off-diagonal elements of $Q(\mu)$ are nonpositive and independent of μ. It follows that $Q(\mu)$ is nonsingular and moreover $Q^{-1}(\mu) \geqslant 0$ if and only if there exist some $c > 0$ and $x \geqq 0$ such that $Q(\mu)x = c$, where c and x are both $n \times 1$ vectors. See e.g. [5, p. 393]. We further notice that the diagonal elements of $Q(\mu)$ are positive for all $\mu \geqq \hat{\mu}$. Note that, in general, they will *not* be equal to each other.

Second, we observe that if $\mu \geqq \hat{\mu}$ increases, then the diagonal elements of $Q(\mu)$ increase whereas the off-diagonal elements of this matrix do not change. Hence, using the set H, we can easily conclude that $H \neq \emptyset$ (take $x = (1, \dots, 1)'$ and observe that $Q(\mu)x > 0$ if μ is sufficiently high). Notice also that if $\mu \in H$, then $\lambda \in H$ for all $\lambda \geqq \mu$.

Third, consider the case where $\hat{\mu} \notin H$. We observe that in this case $\mu > \hat{\mu}$ if $\mu \in H$.

Using Definition 2, we now present Lemma 3.

Lemma 3.

Consider the matrix $Q(\mu)$, the sets M and H, and the scalar $\hat{\mu}$ of Definition 2. *Assume that $\hat{\mu} \notin H$. Next, define the following*:

- $\mu^* \equiv \inf\{\mu \in M \mid \mu \in H\}$.
- *Let $c > 0$ be a given $n \times 1$ vector. Then we define for $\mu \in H$, the vector $y(\mu) \equiv Q^{-1}(\mu)c$. (Notice that $y(\mu) > 0$.)*
- *Let $\{\alpha_v\}$ be a decreasing sequence of scalars in H such that* $\lim_{v\to\infty} \alpha_v = \mu^*$, *and let $\{y(\alpha_v)\}$ be the corresponding sequence of vectors associated with the vector c as just defined. Then we define $s(v) \equiv \sum_{i=1}^{n} y_i(\alpha_v)$.*

We then have the following:

(a) $\mu^* \notin H$.
(b) *Let $\lambda, \mu \in H$ with $\mu > \lambda$. Then $y(\lambda) > y(\mu)$.*
(c) *$\{y(\alpha_v)\}$ is an increasing sequence of vectors.*
(d) *$\{s(v)\}$ is an increasing sequence and* $\lim_{v\to\infty} s(v) = +\infty$.
(e) *The scalar μ^* satisfies $\mu^* \geqq \hat{\mu}$. Moreover, there exists an $n \times 1$ vector $x^* \geqslant 0$ such that $Q(\mu^*)x^* = 0$.*

Proof.

(a) Suppose $\mu^* \in H$. As a result there exists a vector $x \geqq 0$ such that $Q(\mu^*)x > 0$. Then there is also a scalar $\hat{\mu} < \gamma < \mu^*$ such that $Q(\gamma)x > 0$, which implies that $\gamma \in H$, contradicting the definition of μ^*.
(b) We have $Q(\lambda)y(\lambda) - Q(\mu)y(\mu) = c - c = 0$. Consequently

$$Q(\lambda)[y(\lambda) - y(\mu)] = Q(\mu)y(\mu) - Q(\lambda)y(\mu) = [Q(\mu) - Q(\lambda)]y(\mu).$$

Because $\lambda, \mu \in H$, we have $Q^{-1}(\lambda) \geqslant 0$. Since moreover $\mu > \lambda > \mu^*$, it follows that

$$y(\lambda) - y(\mu) = Q^{-1}(\lambda)[Q(\mu) - Q(\lambda)]y(\mu) > 0,$$

which completes the proof.

(c) Follows directly from part (b) of this lemma.

(d) From part (c) of this lemma and the definition of $s(v)$, it follows that $\{s(v)\}$ is an increasing sequence. Suppose that the sequence is bounded from above. Then $y(\alpha_v)$ must be bounded from above as well since $y(\alpha_v) > 0$. Hence, $\lim_{v\to\infty} y(\alpha_v) = \bar{y} > 0$, say. Consider now $Q(\alpha_v)y(\alpha_v) = c$. Taking $v \to \infty$, we obtain $Q(\mu^*)\bar{y} = c$, where $c > 0$ and $\bar{y} > 0$. This means that $\mu^* \in H$, which contradicts part (a) of this lemma.

(e) Since it is assumed that $\hat{\mu} \notin H$, it follows that $\mu^* \geqq \hat{\mu}$. Next, define $x(v) = y(\alpha_v)/s(v)$. Observe that $x(v) \in S = \left\{z \geqslant 0 \mid \sum_{i=1}^n z_i = 1\right\}$, and that S is a compact set. Hence, we may assume, without loosing any generality, that $\{x(v)\}$ converges to some $x^* \in S$ with $x^* \geqslant 0$ if $v \to \infty$. Further, there holds $Q(\alpha_v)x(v) = c/s(v)$. If we let $v \to \infty$, then $\alpha_v \to \mu^*$, and $c/s(v) \to 0$ as a result of part (d) of this lemma. Thus, in the limit we obtain $Q(\mu^*)x^* = 0$. □

In Section 5 we will use Lemma 3, in particular its part (e). In order to understand this part of the lemma it is useful to examine the special case where (i) $q_{ii} = q > 0$ for all i, $\mu_{\min} = -1$, $\hat{\mu} = 0$ and $p_i(\mu) = \mu$ for all i. In that case, we can write $Q = qI - Q^+$, and hence $Q(\mu) = (\mu + q)I - Q^+$, where Q^+ is a nonnegative matrix. Suppose further that (ii) it is *not* true that $(qI - Q^+)^{-1} \geqslant 0$. Applying part (e) of Lemma 3, we can state that (i) and (ii) imply that there exists a real number $\mu^* \geqq 0$ and an $n \times 1$ vector $x^* \geqslant 0$, such that $((\mu^* + q)I - Q^+)x^* = 0$. In fact, this means that under (i) and (ii), the matrix Q^+ must have a Perron root which is greater than or equal to q. We remark that this is a well-known result, see e.g. [6, pp. 18–21]. Concluding, we see that part (e) of Lemma 3 generalizes this known result (for matrices with *identical* diagonal elements and nonpositive off-diagonal elements) to our matrix $Q(\mu)$ with nonidentical diagonal entries and more general functions $p_i(\mu)$. We remark that the results derived here also generalize related results given in [4].

4. Stability results for the matrix *C*

In this section we give a number of stability theorems for the matrix C of (1). In all cases considered, either A or B is a negative diagonal matrix.

Theorem 1. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} < 0$ and $b_{ii} < 0$ for all i. Then we have*:

(a) *if* $n = 1$, *then C is stable*;
(b) *if* $n = 2$, $b_{ij} = 0$ *for all* $i \neq j$, *and A is stable*, *then C is stable*;
(c) *if* $n = 2$, $a_{ij} = 0$ *for all* $i \neq j$, *and B is stable*, *then C is stable.*

Proof. The proof can be established by verifying that under the conditions given, the characteristic equation of the matrix C satisfies the so-called modified Routh-Hurwitz conditions, see [2, p. 92]. The calculations involved are straightforward and can be left to the reader. □

In order to show that part (b) of the theorem cannot be generalized to the case $n > 2$, we give the following counterexample for the case $n = 3$:

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -3 & 2 \\ 2 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}. \tag{4}$$

The matrix A is stable, as its eigenvalues are given by $-0.028 \pm 0.820\mathrm{i}$ and -5.95. The eigenvalues of the corresponding matrix C are $0.060 \pm 1.247\mathrm{i}$, $-0.101 \pm 0.202\mathrm{i}$, -0.021 and -5.896. So, this matrix C is unstable.

The following counterexample demonstrates that part (c) of the theorem cannot be generalized to the case $n > 2$ either:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 2 & 1 \\ 0 & -4 & 3 \\ 2 & 1 & -4 \end{bmatrix}. \tag{5}$$

The matrix B is a stable matrix, with eigenvalues $-5.500 \pm 0.323\mathrm{i}$ and -1.000. The eigenvalues of the corresponding matrix C are $0.012 \pm 2.367\mathrm{i}$, $-0.271 \pm 0.973\mathrm{i}$ and $-0.592 \pm 2.291\mathrm{i}$, which shows that this matrix C is unstable.

The next theorem gives a sufficient stability condition for the matrix C for arbitrary values of n in case B is a negative diagonal matrix. It states that in this case C is stable if the matrix A has an ndd.

Theorem 2. *Consider real* $n \times n$ *matrices A and B and the* $2n \times 2n$ *matrix C as given in* (1). *Assume that* $a_{ii} < 0$ *and* $b_{ii} < 0$ *for all i. If, in addition,* $b_{ij} = 0$ *for all* $i \neq j$ *and A has an ndd, then C is stable.*

Proof. From part (b) of Lemma 2 we conclude that we only have to demonstrate that C cannot have a pair of eigenvalues $\mu_{1,2} = \alpha \pm \beta\mathrm{i}$ with $\alpha \geqq 0$ and $\beta > 0$. In order to do so, let us suppose that C has such a pair of eigenvalues. Limiting the attention to the eigenvalue μ_1 (the case of μ_2 can be treated analogously), it then follows from part (a) of Lemma 2 that the matrix $T(\mu_1) \equiv (t_{ij}^{\mu_1})$, defined by $T(\mu_1) = (\mu_1 I - A - (1/\mu_1)B)$, satisfies $\det(T(\mu_1)) = 0$. Thus, $T(\mu_1)$ has an eigenvalue equal to zero. As a result, the matrix $(-T(\mu_1))$ has an eigenvalue equal to zero as well, which implies that $(-T(\mu_1))$ is not a stable matrix.

We next observe that the off-diagonal elements of $(-T(\mu_1))$ are identical to the corresponding off-diagonal elements of the matrix A. Furthermore, the real part of the diagonal element $(-t_{ii}^{\mu_1})$ of $(-T(\mu_1))$ satisfies $\mathrm{Re}(-t_{ii}^{\mu_1}) = -\alpha + a_{ii} + (b_{ii}\alpha/(\alpha^2+\beta^2)) \leqq a_{ii}$, for all i. Because A has an ndd, we conclude that the matrix which is obtained from $(-T(\mu_1))$ by replacing its diagonal element $(-t_{ii}^{\mu_1})$ by its real part $\mathrm{Re}(-t_{ii}^{\mu_1})$, $i = 1, \ldots, n$, has an ndd as well. In turn, this implies that $(-T(\mu_1))$ is a stable matrix, see [1, p. 141]. We have obtained a contradiction, which establishes the proof. $\square$

We make two remarks with respect to Theorem 2. First, in [3, Corollary 2] we have shown that if (i) A has an ndd, (ii) B is a negative diagonal matrix, (iii) $a_{ii}^2 \geqq 2|b_{ii}|$ for all i, and (iv) there exist positive scalars $m_1, \ldots, m_n$ such that

$$m_i\sqrt{a_{ii}^2 + 2b_{ii}} > \sum_{j \neq i} m_j |a_{ij}|$$

for all i, this is sufficient for stability of C. We observe now that the above Theorem 2 is a much stronger result, i.e., it turns out that the conditions (iii) and (iv) are not needed at all. Second, we cannot state the counterpart result that C is stable in case A is a negative diagonal matrix and B has an ndd. A counterexample is provided again by (5).

In the next theorem we still consider the situation in which B is a negative diagonal matrix. We suppose now furthermore that all diagonal elements of B are equal. We show that in this case a necessary and sufficient condition for stability of the matrix C is that the matrix A is stable.

Theorem 3. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} < 0$ and $b_{ii} = b < 0$ for all i, and $b_{ij} = 0$ for all $i \neq j$. Then C is stable if and only if A is stable.*

Proof. From part (a) of Lemma 2 we know that an eigenvalue μ of C is a root of $\det(\mu^2 I - \mu A - bI) = \det((\mu^2 - b)I - \mu A) = 0$. Note that $\mu = 0$ is not a root, so we can concentrate on finding the roots of $\det(((\mu^2 - b)/\mu)I - A) = 0$. Clearly, there is a straightforward relation between an eigenvalue, say λ, of A and any nonzero eigenvalue μ of C: $\lambda = (\mu^2 - b)/\mu$. Let $\lambda = \rho + \gamma\mathrm{i}$ and $\mu = \alpha + \beta\mathrm{i}$. What we have to prove now is that (i) $\rho < 0$ implies $\alpha < 0$ and (ii) $\rho \geqq 0$ implies $\alpha \geqq 0$. In order to do so we work out the relation between λ and μ:

$$\begin{aligned}
&\mu^2 - b = \lambda\mu \\
&\quad \Leftrightarrow (\alpha + \beta\mathrm{i})^2 - b = (\rho + \gamma\mathrm{i})(\alpha + \beta\mathrm{i}) \\
&\quad \Leftrightarrow (\alpha^2 - \beta^2 + 2\alpha\beta\mathrm{i}) - b = \rho\alpha - \gamma\beta + (\rho\beta + \gamma\alpha)\mathrm{i} \\
&\quad \Leftrightarrow \begin{cases} \alpha^2 - \beta^2 - b = \rho\alpha - \gamma\beta & (*) \\ 2\alpha\beta = \rho\beta + \gamma\alpha & (**). \end{cases}
\end{aligned}$$

First consider the case $2\alpha = \rho$. For this case we have that both (i) and (ii) above hold. Next, consider the case $2\alpha \neq \rho$. For this case we obtain from $(**)$ that $\beta = \gamma\alpha/(2\alpha - \rho)$. Using this in $(*)$ we find:

$$\alpha^2(2\alpha-\rho)^2 - (\gamma\alpha)^2 - b(2\alpha-\rho)^2 = \rho\alpha(2\alpha-\rho)^2 - \gamma^2\alpha(2\alpha-\rho).$$

Rearranging terms we write this expression as

$$4\alpha^4 - 8\rho\alpha^3 + (5\rho^2 + \gamma^2 - 4b)\alpha^2 - (\rho^3 + \gamma^2\rho - 4b\rho)\alpha - b\rho^2 = 0,$$

i.e., a polynomial $\sum_{i=0}^{4} c_i\alpha^i = 0$. In case $\rho < 0$, then $c_i > 0$ for all $i = 0, \ldots, 4$. (Remember that $b < 0$.) Hence any real root of the polynomial is negative, i.e., $\alpha < 0$. This proves (i) above. In case $\rho > 0$, then $c_1 < 0$ and $c_3 < 0$, whereas $c_0 > 0$, $c_2 > 0$, and $c_4 > 0$. Hence any real root of the polynomial is positive in this case, i.e., $\alpha > 0$. Finally, in case $\rho = 0$, then $c_0 = c_1 = c_3 = 0$ and the polynomial reduces to $4\alpha^4 + (\gamma^2 - 4b)\alpha^2 = 0$. The only (real) root of this polynomial is $\alpha = 0$. This proves (ii) above. □

In order to show that the assumption that all diagonal elements of the matrix B are equal is essential for the 'only if' part of Theorem 3, we give the following example:

$$A = \begin{bmatrix} -1 & 0.9 & 0.2 \\ 0.6 & -1.5 & 0.9 \\ 1 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{6}$$

The eigenvalues of A read 0.042, −1.943 and −2.600, which means that A is an unstable matrix. The eigenvalues of the corresponding matrix C are $-0.009 \pm 1.281\mathrm{i}$, $-0.991 \pm 0.956\mathrm{i}$, −0.725 and −1.774. Thus, C is stable. Analogously, a counterexample for the 'if part' is given again by (4).

As the counterpart of Theorem 3 we now present a theorem for the case in which A is a negative diagonal matrix with all diagonal elements equal. We remark that the proofs of the Theorems 3 and 4 are related.

Theorem 4. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} = a < 0$ and $b_{ii} < 0$ for all i, and $a_{ij} = 0$ for all $i \neq j$. If B is stable and, in addition, $|a| > |\gamma|/\sqrt{-\rho}$ for all eigenvalues $\lambda = \rho + \gamma\mathrm{i}$ of B, then C is stable.*

Proof. Using part (a) of Lemma 2 we find that an eigenvalue $\mu = \alpha + \beta\mathrm{i}$ of C is a root of $\det(\mu(\mu - a)I - B) = 0$. Let $\lambda = \rho + \gamma\mathrm{i}$ be an eigenvalue of B and we obtain $\lambda = \mu(\mu - a)$. We have to show that if $\rho < 0$ and $|a| > |\gamma|/\sqrt{-\rho}$, then $\alpha < 0$. This can be done as follows:

$$\begin{aligned} &\lambda = \mu(\mu - a) \\ &\quad \Leftrightarrow \rho + \gamma\mathrm{i} = (\alpha + \beta\mathrm{i})(\alpha - a + \beta\mathrm{i}) = \alpha(\alpha - a) - \beta^2 + (2\alpha - a)\beta\mathrm{i} \\ &\quad \Leftrightarrow \begin{cases} \rho = \alpha(\alpha - a) - \beta^2 & (*) \\ \gamma = (2\alpha - a)\beta & (**). \end{cases} \end{aligned}$$

First suppose that $2\alpha - a = 0$. We then have $\alpha = \frac{1}{2}a < 0$, i.e., C is stable. Next, we consider the case $2\alpha - a \neq 0$. For this case we obtain from $(**)$ that $\beta = \gamma/(2\alpha - a)$. Using this in $(*)$ we find

$$\rho(2\alpha - a)^2 = \alpha(\alpha - a)(2\alpha - a)^2 - \gamma^2.$$

We can write this expression as

$$4\alpha^4 - 8a\alpha^3 + (5a^2 - 4\rho)\alpha^2 - (a^3 - 4\rho a)\alpha - (\gamma^2 + \rho a^2) = 0,$$

i.e., a polynomial $\sum_{i=0}^{4} c_i\alpha^i = 0$. Since $a < 0$ and $\rho < 0$, we obtain that $c_i > 0$ for $i = 1, \ldots, 4$. From the assumption that $|a| > |\gamma|/\sqrt{-\rho}$, it follows that c_0 is positive as well, hence this polynomial has only negative real roots for α, and we are done. □

We recall that [3, Corollary 1] also gives a stability result for the matrix C for the situation in which A is a negative diagonal matrix. In particular, it is shown there that if (i) A is a negative diagonal matrix, (ii) B has an ndd, and (iii) the diagonal elements of A and B are such that $a_{ii}^2 \geqq 2|b_{ii}|$ for all i, then this is sufficient to establish stability of C. The following matrices give an example in which we cannot apply that stability result, because the conditions (ii) and (iii) are not satisfied:

$$A = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & -1 \\ 2 & 0 & -1 \end{bmatrix}. \tag{7}$$

However, with the eigenvalues of matrix B given by $-0.365 \pm 0.692\mathrm{i}$ and -3.267, these matrices A and B satisfy all requirements of Theorem 4, which implies that the associated matrix C must be stable (notice that $0.692/\sqrt{0.365} = 1.14 < 1.5$). This is confirmed by the eigenvalues of C, which are equal to $-0.073 \pm 0.511\mathrm{i}$, $-0.750 \pm 1.645\mathrm{i}$ and $-1.427 \pm 0.511\mathrm{i}$.

The following corollary is an immediate consequence of Theorem 4.

Corollary 1. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} = a < 0$ and $b_{ii} < 0$ for all i, and $a_{ij} = 0$ for all $i \neq j$. If B is stable and has only real eigenvalues, then C is stable.*

5. Instability results for the matrix C

In this section we present two theorems that give sufficient conditions for instability of the matrix C of (1) in case either A or B is a negative diagonal matrix whereas the other matrix is Metzlerian. In particular, we demonstrate that under the conditions given below, C has a real nonnegative eigenvalue. As a byproduct we also show that these nonnegative eigenvalues are accompanied by a semipositive eigenvector. Before presenting the theorems, we recall that a Metzlerian matrix has no ndd if and only if it is unstable. The first theorem addresses the case in which A is a negative diagonal matrix and B is a Metzlerian matrix.

Theorem 5. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} < 0$ and $b_{ii} < 0$ for all i, and $a_{ij} = 0$ and $b_{ij} \geqq 0$ for all $i \neq j$. Assume that B has no ndd. Then C has an eigenvalue $\mu^* \geqq 0$ with a corresponding semipositive eigenvector.*

Proof. First, in order to be able to apply part (e) of Lemma 3, take for $\mu_{\min}$ some arbitrary negative real number, and define the functions $p_i(\mu) = \mu^2 - \mu a_{ii}$ for all $\mu \in (\mu_{\min}, \infty)$ and $i = 1, \ldots, n$. Next, define $\hat{\mu} = 0$ and $M = [\hat{\mu}, \infty)$ and notice that $p_i(\mu) \geqq 0$ as well as $p_i'(\mu) > 0$ for all $\mu \in M$ and all i. Next, define for $\mu \in M$ the matrix $Q(\mu) = -B + P(\mu)$, where $P(\mu) = \operatorname{diag}(p_1(\mu), \ldots, p_n(\mu))$, and the set $H = \{\mu \mid Q^{-1}(\mu) \geqslant 0\}$. Notice that $Q(\mu) = \mu^2 I - \mu A - B$ for all $\mu \in M$, and that $Q(0) = -B$.

Next, we observe by using [5, p. 393] that because B has no ndd, we must have either case (a) in which the matrix $(-B)^{-1}$ exists and has at least one negative element, or case (b) in which B is a singular matrix.

Consider first case (a). In this case the inverse of the matrix $Q(0)$ exists and has at least one negative element. As a result $0 \notin H$, and we conclude from part (e) of Lemma 3 that there exists a scalar $\mu^* \geqq 0$ and an $n \times 1$ vector $r^* \geqslant 0$ such that $Q(\mu^*)r^* = 0$. We notice that in fact $\mu^* > 0$ in this case, because $\mu^* = 0$ would mean that $Br^* = 0$, which contradicts the present assumption that B is a nonsingular matrix. Proceeding, from part (a) of Lemma 2 it follows that μ^* is an eigenvalue of C. Furthermore, we observe that the $2n \times 1$ vector $((\mu^* r^*)', (r^*)')' \geqslant 0$ is an eigenvector of C associated with the eigenvalue μ^*.

Finally, let us turn to case (b) in which the matrix B is singular. In this case the inverse of the matrix $Q(0)$ does not exist. As a result $0 \notin H$, and again we can apply part (e) of Lemma 3 to conclude that there exists a scalar $\mu^* \geqq 0$ and an $n \times 1$ vector $r^* \geqslant 0$ such that $Q(\mu^*)r^* = 0$. The remainder of the proof follows as under case (a). $\square$

The final theorem is the counterpart of Theorem 5 and considers the case in which B is a negative diagonal matrix and A is a Metzlerian matrix.

Theorem 6. *Consider real $n \times n$ matrices A and B and the $2n \times 2n$ matrix C as given in* (1). *Assume that $a_{ii} < 0$ and $b_{ii} < 0$ for all i, and $a_{ij} \geqq 0$ and $b_{ij} = 0$ for all $i \neq j$. Assume that $A + D$ has no ndd, where $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$, $d_{ii} = ((b_{ii}/d) - d) < 0$ for $i = 1, \ldots, n$, and $d = \max_i \sqrt{-b_{ii}} > 0$. Then C has an eigenvalue $\mu^* \geqq d > 0$ with a corresponding semipositive eigenvector.*

Proof. We know from part (a) of Lemma 2 that μ is an eigenvalue of C if and only if $\det(\mu^2 I - \mu A - B) = 0$. Remark that $\mu = 0$ cannot be an eigenvalue of C because B is a nonsingular matrix. So, μ is an eigenvalue of C if and only $\det(-A + (\mu I - (1/\mu)B)) = 0$.

In order to apply part (e) of Lemma 3, take $\mu_{\min} = 0$ and define the functions $p_i(\mu) = \mu - (b_{ii}/\mu)$ for all $i = 1, \ldots, n$ and all $\mu \in (0, \infty)$. Next, define $\hat{\mu} = \max_i \sqrt{-b_{ii}} > 0$ and the set $M = [\hat{\mu}, \infty)$. We then see that $p_i(\mu) \geqq 0$ as well as $p_i'(\mu) > 0$ for all $\mu \in M$. Finally, define the matrix $Q(\mu) = -A + P(\mu)$, with $P(\mu) = \mathrm{diag}(p_1(\mu), \ldots, p_n(\mu))$, and the set $H = \{\mu \in M \mid Q^{-1}(\mu) \geqslant 0\}$. We remark that $\hat{\mu} = d$ and $P(\hat{\mu}) = P(d) = -D$. Because the matrix $A + D$ has no ndd, it follows that the matrix $(-Q(d))$ has no ndd. Using [5, p. 393] we can conclude that $d \notin H$. Part (e) of Lemma 3 then again easily establishes the proof. $\square$

Concluding this section, we remark that from Theorem 5 we learn that the matrix C is unstable in case A is a negative diagonal matrix and B is a Metzlerian matrix that does not possess an ndd. Using the definition of a dominant diagonal number of a matrix as introduced in Section 2, we observe that the fact that B has no ndd means that the dominant diagonal number σ_B of B satisfies $\sigma_B \geqq 1$. Analogously, we see from Theorem 6 that C is unstable in case B is a negative diagonal matrix, A is a Metzlerian matrix, and $A + D$ has no ndd, where D is a certain negative diagonal matrix. Using (b) of Lemma 1, we observe that the latter implies that the dominant diagonal number σ_A of A satisfies $\sigma_A > 1$ (in fact, it can be verified that in this case we have $\sigma_A \geqq \min_i\{1 + (d_{ii}/a_{ii})\} = \min_i\{1 + (b_{ii}/da_{ii}) - (d/a_{ii})\} > 1$, cf. the proof of (b) of Lemma 1). Recall that in the stability theorems in [3] the matrices A and B are required to have an ndd with a deliberately chosen strength factor τ that is 'sufficiently smaller' than unity, which implies that $\sigma_A < 1$ and $\sigma_B < 1$. These observations thus unify the instability theorems of this section and relate them to the stability theorems of [3].

6. Conclusions

In this paper we considered stability properties of the matrix C given in (1), for the case in which either the matrix A or the matrix B is a negative diagonal matrix. In Section 2 we introduced the concept of the dominant diagonal number of a matrix, and in Section 3 we gave two useful lemmas. In Section 4 we presented four theorems that give sufficient conditions for stability of C in terms of properties of A and B. We have seen that these theorems are stronger than and/or extend related stability results of [3]. In Section 5 we presented two theorems which give conditions such that C has a real nonnegative eigenvalue, which implies that C is an unstable matrix. We have also seen that these eigenvalues have a corresponding semipositive eigenvector. Finally, we pointed out that we can unify the instability theorems and relate them to the results of [3] by using the concept of a dominant diagonal number of a matrix.

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